RESEARCH PROPOSAL FOR P. DELIGNE CONTEST

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Over the recent decades representation theory has gained a central role in modern mathematics, linking such areas as number theory, topology, differential equations, algebraic and arithmetic geometry, theory of automorphic forms. A cornerstone of modern representation theory is the Langlands program of classification of admissible representations of reductive groups.

The geometric interpretation of Langlands program for groups over function fields is been developed since 1970s. This approach is based on two key constructions due to Drinfeld and Beilinson. The first construction is originally introduced by Drinfeld for the group $GL(2)$; it originates from a generalization of the Lang-Rosenlicht geometric classical (Abelian) class field theory to the case of non-commutative group. The second construction is based on the Beilinson-Bernstein procedure of localization of $\mathfrak{g}$-modules on generalized flag manifold G/B of arbitrary semisimple Lie group G. In order to establish an equivalence of these two basic constructions a concept of chiral Hecke algebra was introduced, explicit construction of which is still to be determined.

One possible approach is based on a construction of distinguished coordinates (also referred to as separated variables) on G-orbits. Such coordinates should have a distinct group-theoretic meaning; they should make possible to obtain integral representations of a standard set of special functions on G. In this Proposal I present several series of my results obtained in this framework over the last 8 years. At the end of Proposal I outline the problems yet to be solved.

§1. Geometric interpretation of isomonodromy method. Papers [O1] and [O2], which contain the central part of my Ph.D. thesis, are devoted to studying the isomonodromy method for Fuchsian differential equations of order two on $\mathbb{CP}^1$ in the setting of the first Drinfeld's construction. In particular, [O1], [O2] generalize the results of [AL] to the case of arbitrary number of singularities of equation. Let $\mathcal{M}_n(\lambda_1,\ldots,\lambda_n)$ be the coarse moduli space of collections $(\mathcal{L},\,\nabla,\,\phi;\,\lambda_1,...,\lambda_n)$, on $X=\mathbb{P}^1\setminus\{a_1,\ldots,a_n\}$, where $\mathcal{L}$ is a rank 2 bundle on $\mathbb{P}^1$ equipped with a connection $\nabla:\mathcal{L}\to\mathcal{L}\otimes\Omega^1_{\mathbb{P}^1}(\mathfrak{M})$ with $\mathfrak{M}=a_1+...+a_n$, and the horizontal isomorphism $\phi:\det\mathcal{L}\simeq\mathcal{O}_{\mathbb{P}^1}$; the eigenvalues of the residues $\mathrm{Res}_{\,a_i}\nabla$ are $(\lambda_i,\,-\lambda_i)$.

• Bi-rational isomorphisms between moduli spaces $\mathcal{M}_n(\lambda_1,\ldots,\lambda_n)$ and $\mathcal{M}_n(\tilde\lambda_1,\ldots,\tilde\lambda_n)$ for certain $\tilde\lambda_i$ are given by the elementary modifications of the $SL(2)$ bundle $\mathcal{L}$ performed at marked points $x_1,\ldots,x_n$. In [O2] the group structure of such isomorphisms is calculated.

Theorem 1. *The group of isomorphisms between moduli spaces $\mathcal{M}_n$ is isomorphic to the affine Weyl group of type C_n.*

• This result generalizes classical results of Gauss, Kummer and Heun on certain relations between hypergeometric and Heun functions which correspond to particular case $n=3,4$.

Corollary 1. *(i) The 24 Kummer series of hypergeometric function admits a transitive action of bi-octaedron group $W(C_3)\simeq\big(\mathbb{Z}/2\mathbb{Z}\big)\rtimes\mathfrak{S}_4$ of order 48. The Gauss relations between hypergeometric functions represent the translation part of affine Weyl group $W(\widehat{C}_3)$.*
(ii) The 192 Kummer-type series of Heun functions admit a transitive action of finite Weyl group $W(C_4)=\big(\mathbb{Z}/2\mathbb{Z}\big)^4\rtimes\mathfrak{S}_4$ of order 384. Gauss-type relations for Heun functions represent the translation part of affine Weyl group $W(\widehat{C}_4)$.

• In [O1] the distinguished parametrization of moduli space $\mathcal{M}_n$ is constructed; besides, this establishes a geometric meaning of isomonodromy method.

Theorem 2. [O1] *(i) The open part of $\mathcal{M}_n$ is isomorphic to the open part of moduli space of exact sequences*

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \longrightarrow \widetilde{\mathcal{L}} \longrightarrow \bigoplus_{k=1}^{n-3} \delta_{x_k} \otimes p_k \otimes \mathcal{T}_{x_k} \longrightarrow 0$$

where $\widetilde{\mathcal{L}}$ is a vector bundle on $\mathbb{P}^1$ of degree -1 with a logarithmic connection $\widetilde{\nabla}$ with fixed eigenvalues of residues at a_i: $(1-\lambda_1, \lambda_1)$ and $(\lambda_i, -\lambda_i)$ for $1 < i \leq n$; here δ_{x_i} denotes the sky-scraper sheaf supported at $x_k \in X$ and $p_k \subset \widetilde{\mathcal{L}}|_{x_k}$ is a one-dimensional subspace.
(ii) The open part of the complete self-intersection locus Θ_n of compactifying divisor $\overline{\mathcal{M}_n} \setminus \mathcal{M}_n$ has dimension $n-3$, and it is isomorphic to the affine space of certain logarithmic connections ∇_{Θ_n} in the rank two bundle $\mathcal{O} \oplus \mathcal{T}(-\mathfrak{M})$ with fixed isomorphism $\det \mathcal{O} \oplus \mathcal{T}(-\mathfrak{M}) \simeq \mathcal{O}(-a_1 - \ldots - a_{n-2})$. The connection ∇_{Θ_n} has fixed eigenvalues of the residues at simple poles a_i: they are $(1-\lambda_i, \lambda_i)$ for $i = 1, \ldots, n-2$ and $(\lambda_i, -\lambda_i)$ for $i = n-1, n$.

Besides, it is shown [O1] that the coordinates (x_k, p_k) are exactly the parameters of apparent singularities [AB] of the system of linear differential equations defined by the $sl(2)$-connection ∇, and thus they are the canonical dynamical variables of the corresponding isomonodromy problem.

• In [O1] the following geometric interpretation of isomonodromy method is given. As it been explained, isomonodromy dynamics modifies the spectral data of the connection ∇; precisely, it deforms the spectral curve C_λ of ∇ inside $\mathcal{M}_n$. It is shown [O1] that under the isomonodromy dynamics C_λ tends to a limit cycle (degenerate spectral curve) which coincides with the compactifying divisor in Drinfeld's compactification of $\mathcal{M}_n$.

§2. Monopole spaces and Quantum groups. In a series of papers [GKLO1]-[GKLO2] an explicit interrelation between the separation of variables and representation theory of Poisson-Lie groups is established. More precisely, paper [GKLO1] is devoted to identification of moduli space of holomorphic maps from a rational curve to the generalized flag variety G/B of a semisimple complex Lie group G with certain symplectic orbits of the Yangian $Y(\mathfrak{g})$ (see [D2]).

• Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank ℓ with Cartan matrix $\|a_{i,j}\|$, and let $\mathfrak{b}$ be its Borel subalgebra; denote by Y_{cl} the Poisson-Lie group (classical limit) of the Yangian. In theorem 3.1 in [GKLO1] a new class of representations of $Y(\mathfrak{g})$ and $Y(\mathfrak{b})$ is constructed. These representations turn out to quantize (in the sense of [D2]) the moduli space of G-monopoles of topological charge $(m_1, \ldots, m_\ell)$; the precise statement is as follows.

Theorem 3. *(i) The open parts $\mathcal{O}^{(0)}$ of the rational symplectic leaves of $Y_{cl}(\mathfrak{b})$ corresponding to the representations constructed in theorem 3.1 (*[GKLO1]*) are isomorphic to the open parts of the spaces of the based maps*

$$(\mathbb{P}^1, \infty) \to (G/B, B_+)$$

of the fixed multi-degree $\mathbf{m} = (m_1, \cdots, m_\ell) \in H_2(G/B, \mathbb{Z})$.

(ii) The open parts $\mathcal{O}^{(0)}$ of the rational symplectic leaves of $Y_{cl}(\mathfrak{g})$ corresponding to the representations constructed in theorem 3.1 ([GKLO1]*) are isomorphic to the spaces of the based maps with additional restrictions $\sum_{j=1}^{\ell} m_j a_{ji} = l_i \in \mathbb{Z}_+$.*

• In [GKLO2] the class of representations proposed in Theorem 3.1 [GKLO1] is generalized to a class of the infinite-dimensional representations of the finite-dimensional quantum groups $U_q(\mathfrak{g})$ and affine quantum groups $U_q(\hat{\mathfrak{g}})_{c=0}$ for an arbitrary semi-simple Lie algebra $\mathfrak{g}$.

• Similar to the connection of the Yangian representations with the quantization of the monopoles on $\mathbb{R}^3$ the proposed representations of the affine algebra turn out to be connected with the quantization of the periodic monopoles on $\mathbb{R}^2 \times S^1$. Thus the classification of the trigonometric r-matrices underlying the quantum affine algebras $U_q(\hat{\mathfrak{g}})$ [BD] corresponds to the classification of the particular class of asymptotic boundary conditions on a monopole solutions on $\mathbb{R}^2 \times S^1$. It is natural to make one step further and consider the quantization of the moduli space of the double-periodic monopoles on $\mathbb{R} \times S^1 \times S^1$. Presumably this should correspond to the quantum elliptic algebras and the choice of asymptotic boundary conditions may be associated with elliptic r-matrix [BD].

§3. Baxter operators and Representation theory. The notion of Q-operator was introduced by Baxter as a key tool to solve quantum integrable systems [B]. These operators were originally constructed for a particular class of integrable systems associated to affine Lie algebras $\widehat{\mathfrak{gl}}_N$. Later a new class of Q-operators corresponding to $\widehat{\mathfrak{gl}}_N$-Toda chains was proposed [PG].

The series of papers [GKLO3], [GLO1]-[GLO4] is devoted to studying of Baxter's Q-operator formalism [B] for the quantum Toda chain associated to a reductive group G; a special interest to Toda chain is due to an identification [K] of $\mathfrak{g}$-Toda wave function with $\mathfrak{g}$-Whittaker function. In particular, an application of Baxter Q-operator formalism provides a set of recursive properties of Whittaker functions and this leads to new results on automorphic L-functions.

• In 1996 Givental discovered a remarkable integral representation for the $GL(N)$ Toda wave function. In [GKLO3] a group-theoretic description of the Givental integral formula is given and the corresponding representation of the universal enveloping algebra $U\mathfrak{gl}(N)$ is constructed (see Proposition 2.1). Besides, it is shown [GKLO3] that in group theory interpretation the contour of integration in Givental formula can be naturally identified with the subset of totally positive unipotent elements $N_+^\circ \subset N_+$.

• One of the most important properties of Givental representation is its recursive structure.

Theorem 4. [GKLO3] *Let $\Psi^{(N-1)}_{\lambda_1,\dots,\lambda_{N-1}}(T_{N-1,1},\dots,T_{N-1,N-1})$ be the $GL(N-1)$ Whittaker function. Then the function*

$$\Psi^{GL(N)}_{\lambda_1,\dots,\lambda_N}(T_{N,1},\dots,T_{N,N}) = (\mathcal{Q}^{GL(N)}_{GL(N-1)} * \Psi^{(N-1)}_{\lambda_1,\dots,\lambda_{N-1}})(T_{N,1},\dots,T_{N,N})$$

is the $GL(N)$ Whittaker function depending on coordinates $T_{N,1},\dots,T_{N,N}$. The recursive operator $\mathcal{Q}^{GL(N)}_{GL(N-1)}$ is an integral operator acting on Whittaker function by a convolution.

In [GKLO3] a natural connection between operators $\mathcal{Q}^{GL(N)}_{GL(N-1)}$ and Pasquier-Gaudin's Q-operators [PG] was discovered.

• A generalization of [GKLO3] to the other Lie algebras is established in [GLO1]-[GLO3]. Namely, [GLO1]-[GLO2] contain explicit formulas for Baxter Q-operators for classical finite and infinite dimensional, and (twisted) affine Lie algebras. Besides, in [GLO3] the Givental integral representation of Whittaker function is generalized to the case of Lie algebras of classical type.

For example, in the case C_n the Givental representation reads as follows.

Theorem 5. *The eigenfunction for C_n open Toda chain is given by*

$$\Psi^{C_n}_{\lambda_1,\dots,\lambda_n}(z_1,\dots,z_n) = \int \prod_{1\le i\le k\le n-1} dz_{k,i} \prod_{k=1}^{n-1} Q^{C_k}_{C_{k+1}}(z_{k+1,1},\dots,z_{k+1,k+1};\, z_{k,1},\dots,z_{k,k}),$$

where $z_i := z_{n,i}$ and the kernels $Q^{C_k}_{C_{k+1}}$ of the integral operators are given by the convolutions of the kernels $Q^{D_{k+1}}_{C_{k+1}}$ and $Q^{C_k}_{D_{k+1}}$:

$$Q^{C_k}_{C_{k+1}}(\underline{z}_{k+1};\underline{z}_k) = \int \prod_{i=1}^{k+1} dx_{k,i}\; Q^{D_{k+1}}_{C_{k+1}}(\underline{z}_{k+1};\underline{x}_{k+1}) \cdot Q^{C_k}_{D_{k+1}}(\underline{x}_{k+1};\underline{z}_k)$$

Observe that the obtained formulas have an interesting recursive structure. Recall that in the case $GL(N)$ (see [G], [GKLO3]) the integral operator $\mathcal{Q}^{GL(N)}_{GL(N-1)}$ intertwining $GL(N-1)$ and $GL(N)$ wave functions has a simple "quasi-classical" structure: it is presented as an exponent of a sum of some (exponential) functions. In the case of other classical groups the integral kernel is given by an integral convolution of two elementary quasi-classical kernels. Also emphasize that elementary quasi-classical kernels $\mathcal{Q}^{C_k}_{D_{k(+1)}}$ intertwine Whittaker models for groups of different types, as $SO(2k)$ and $Sp(2k)$ above.

• In fact [GKL], [GKLO3] there are two (dual) integral representations the $GL(N)$-Whittaker function $\Psi^{GL(N)}_{\underline{\lambda}}(\underline{x})$: Mellin-Barnes representation and Givental representation. Both representations admit recursive structures; Mellin-Barnes representation is recursive with respect to the spectral parameters $\underline{\lambda}=(\lambda_1,\ldots,\lambda_N)$, and Givental representation is recursive with respect to the coordinates $\underline{x}=(x_1,\ldots,x_N)$. This leads to a family of mixed Givental-Mellin-Barnes representations of $GL(N)$ Whittaker functions constructed in [GLO4].

• In this way in [GLO4] two dual Baxter operators are constructed: $\widehat{\mathcal{Q}}(\underline{\gamma},\underline{\lambda}|z)$ and $\mathcal{Q}(\underline{y},\underline{x}|\gamma)$ corresponding to the two integral representations.

Theorem 6. [GLO4] *(i) For the Baxter Q-operator in Givental representation the following relation holds*

$$\int d\underline{x}\;\mathcal{Q}^{GL(N)}(\underline{y},\,\underline{x}|\,\gamma)\,\Psi^{GL(N)}_{\underline{\lambda}}(\underline{x})\;=\;\prod_{i=1}^N\Gamma\Big(\frac{\imath\gamma-\imath\lambda_i}{2}\Big)\;\Psi^{GL(N)}_{\underline{\lambda}}(\underline{y})$$

(ii) For the Baxter Q-operator in Mellin-Barnes representation the following relation holds.

$$\int d\underline{\gamma}\;\widehat{\mathcal{Q}}^{GL(N)}(\underline{\lambda},\underline{\gamma}|z)\,\Psi^{GL(N)}_{\underline{\gamma}_N}(\underline{x})=e^{-e^{(x_N-z)}}\Psi^{GL(N)}_{\underline{\lambda}_N}(\underline{x})$$

• Moreover, in [GLO4] a universal recursive operator for $GL(N)$ Whittaker function been constructed: it is symmetric with respect to both sets of parameters, $\underline{\lambda}$ and $\underline{x}$.

Theorem 7. *The following symmetric recursive relation for $GL(N)$-Whittaker functions holds.*

$$\Psi^{GL(N)}_{\underline{\lambda}}(\underline{x})=e^{\imath\lambda_N x_N}\;\widehat{\mathcal{Q}}(\underline{\lambda},\underline{\lambda}'|\,x_N)*\mathcal{Q}(\underline{x},\underline{x}'|\lambda_N)*\Psi^{GL(N-1)}_{\underline{\lambda}'}(\underline{x}')$$

where $\underline{x}'=(x_1,\ldots,x_{N-1})$ and $\underline{\lambda}'=(\lambda_1,\ldots,\lambda_{N-1})$.

• These results have effective applications to automorphic L-functions. It is shown in [GLO4] that the Stade's relations [St1] between $GL(N)-$ and $GL(N+2)$-Whittaker functions are equivalent to a composition $\mathcal{Q}^{GL(N)}_{GL(N-1)}\circ\mathcal{Q}^{GL(N-1)}_{GL(N-2)}$. Thus the results from [GKLO3]-[GLO4] essentially simplify technical Stade's proofs [St2], [St3] of the Bump and Bump-Freidberg conjectures [Bu], [BuF] on Rankin-Selberg L-functions.

• The main result of [GLO4] established an explicit description of Baxter's Q-operator as an element of spherical Hecke algebra $\mathcal{H}\big(GL(N,\mathbb{R}),\,SO(N,\mathbb{R})\big)$.

Theorem 8. [GLO4] *Let $\phi_{\mathcal{Q}(\lambda)}(g)$ be a K-biinvariant function on $G=GL(N,\mathbb{R})$ given by*

$$\phi_{\mathcal{Q}(\lambda)}(g)=2^N|\det g|^{\imath\lambda}e^{-\pi\,\mathrm{Tr}\,g^tg}$$

(i) Then the action of $\phi_{\mathcal{Q}(\lambda)}$ on Whittaker functions descends to the action of $\mathcal{Q}^{GL(N)}(\lambda)$.
(ii) Then the action of $\phi_{\mathcal{Q}(\lambda)}$ on normalized Whittaker function $\Phi^{GL(N)}_{\underline{\gamma}}(g)$ is given by

$$\big(\phi_{\mathcal{Q}(\lambda)}*\Phi^{GL(N)}_{\underline{\gamma}}\big)(g)=L_\infty(\lambda)\;\Phi^{GL(N)}_{\underline{\gamma}}(g)$$

where $L_\infty(\lambda)$ is the local Archimedean L-factor

$$L_\infty(\lambda)=\prod_{j=1}^N\pi^{-\frac{\imath\lambda-\imath\gamma_j}{2}}\,\Gamma\Big(\frac{\imath\lambda-\imath\gamma_j}{2}\Big)$$

§4. q-deformed Whittaker functions. In [GLO5] the q-Whittaker function is obtained as a certain degeneration of MacDonald polynomial. In [GLO5]-[GLO7] the two dual representations of q-deformation of $GL(N)$-Whittaker function.

• Denote by $\mathcal{P}^{(N)}\subset\mathbb{Z}^{N(N-1)/2}$ a subset of parameters $p_{k,i}$, $1\le i\le k\le N-1$ satisfying the Gelfand-Zetlin conditions $p_{k+1,i}\ge p_{k,i}\ge p_{k+1,i+1}$. Let $\mathcal{P}_{N,N-1}\subset\mathcal{P}^{(N)}$ be a set of $\underline{p}_{N-1}=(p_{N-1,1},\ldots,p_{N-1,N-1})$ satisfying the conditions $p_{N,i}\ge p_{N-1,i}\ge p_{N,i+1}$. Let $q\in\mathbb{R}$ and $q<1$.

Theorem 9. *The q-Whittaker function* ${}^q\Psi_{\underline{z}}(\underline{p}_N)$ *can be written in the following form.*
(I) For $\underline{p}_N$ *being in the dominant domain* $p_{N,1} \geq \ldots \geq p_{NN}$

$$ {}^q\Psi_{\underline{z}}(\underline{p}_N) \,=\, \sum_{\underline{p}_k \in \mathcal{P}^{(N)}} \prod_{k=1}^{N} z_k^{|\underline{p}_k|-|\underline{p}_{k-1}|} \, \frac{\prod\limits_{n=2}^{N-1}\prod\limits_{i=1}^{n-1}(p_{n,i}-p_{n,i+1})_q!}{\prod\limits_{1\leq i\leq n\leq N-1}(p_{n+1,i}-p_{n,i})_q!\,(p_{n,i}-p_{n+1,i+1})_q!}, $$

where we use the notations $|p_n| = \sum_{i=1}^{n} p_{n,i}$ *and* $(n)_q! = (1-q)...(1-q^n)$.
(II) When $\underline{p}_N$ *is outside the dominant domain* ${}^q\Psi_{\underline{z}}(\underline{p}) = 0$.

• In [GLO7] a q-version of Mellin-Barnes representation of $\mathfrak{gl}_N$-Whittaker function is constructed. Namely, given a triangular array of variables $\{z_{k,i};\ 1 \leq i \leq k \leq N\}$ with $z_{N,i} := z_i$, $1 \leq i \leq N$ the following integral formula holds.

Theorem 10. *(I) In the dominant domain* $p_{N,1} \geq \ldots \geq p_{NN}$ *the following holds.*

$$ {}^q\Psi_{\underline{z}_N}(\underline{p}_N) \,=\, \Gamma_q(q)^{\frac{(N-1)(N-2)}{2}} \prod_{\substack{n=1\\ j\leq n}}^{N-1} \oint \frac{dz_{n,j}}{2\pi\imath z_{n,j}} \prod_{1\leq i\leq k\leq N} \Big(\frac{z_{k,i}}{z_{k-1,i}}\Big)^{p_{N,k}} \prod_{n=1}^{N-1} \frac{\prod\limits_{i=1}^{n+1}\prod\limits_{j=1}^{n} \Gamma_q\big(z_{n,j}^{-1}z_{n+1,i}\big)}{n!\,\prod\limits_{j\neq m}\Gamma_q\big(z_{n,m}^{-1}z_{n,j}\big)} $$

where $\Gamma_q(z) = \prod_{n\geq 0}(1-zq^n)^{-1}$ *is a q-version of* Γ*-function.*
(II) When $\underline{p}_N$ *is outside the dominant domain* ${}^q\Psi_{\underline{z}_N}(\underline{p}_N) = 0$.

• Expanding the first formula (from Theorem 9) with respect to parameter $q < 1$ we get

$$ {}^q\Psi_{\underline{z}_N}(\underline{p}_N) \,=\, \mathrm{Tr}_{\mathcal{V}}\, q^{L_0} \prod_{i=1}^{N} e^{\lambda_i E_{ii}} $$

This identity can be viewed as an (Archimedean) q-analog of Shintani-Casselman-Shalika formula for p-adic Whittaker function. In particular, the constructed q-deformed $\mathfrak{gl}_N$-Whittaker function interpolates between Archimedean and p-adic Whittaker functions with respect to parameter q.

• In [GLO7] the function ${}^q\widetilde{\Psi}_{\underline{z}_N}(\underline{p}_N) := {}^q\Psi_{\underline{z}_N}(\underline{p}_N)\prod_{k=1}^{N-1}(p_{N,k}-p_{N,k+1})_q!$ is identified with the character of (finite-dimensional) Demazure module of affine Lie algebra $\widehat{\mathfrak{gl}}_N$.

• In [GLO6] the (q-version of) Mellin-Barnes representation of $\mathfrak{gl}_2$-Whittaker function is realized as a semi-infinite period map. The explicit form of the period map manifests an important role of (q-version of) Γ-function as a topological genus in semi-infinite geometry.

§5. Future research: Archimedean L-functions and Topological field theories. The relevant setting for further development of the relation between the (q-deformed) Whittaker function and semi-infinite cohomology is the topological field theory [W].

• In [GLO9] a functional integral representation for Archimedean L-factors (given by products of Γ-functions) is proposed. The corresponding functional integral arises in the description of type A equivariant topological linear sigma model on a disk. The obtained functional integral representation provides an interpretation of the Γ-function as an equivariant symplectic volume of the infinite-dimensional space of holomorphic maps of the disk to $\mathbb{C}$.

• In [GLO10] it is shown that the Euler integral representation naturally arises as a disk partition function in the equivariant type B topological Landau-Ginzburg model on a disk with the target space $\mathbb{C}$, which is mirror-symmetric to the type A sigma model involved in [GLO9]. These two integral representations of Γ-functions are similar to the two constructions (arithmetic and automorphic) of local Archimedean L-factors. The equivalence of the resulting L-factors is a manifestation of local Archimedean Langlands correspondence.

• In the nearest future I plan to extend the results of [O1], and [GLO4]-[GLO10] to classical Lie algebras, using the results of [GKLO3] and [GLO7]. On this route one may expect a series of interesting and important results in representation theory and arithmetic geometry [P].

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